# Quantum Amplitude Estimation – Applications to Derivative Pricing

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# Classical Monte Carlo

We have seen how classical Monte Carlo can be used to compute the expected payoff $\mu = \mathbb{E}[X]$ of a simple option $X = f(S_T)$:

- Sample $x_1, ..., x_N \overset{i.i.d}{\sim} S_T$,
- Set $\tilde{\mu}_N = \frac{1}{N}\sum_{i=1}^{N} f(x_i)$.

By the law of large numbers,

$$\tilde{\mu}_N \overset{\text{a.s.}}{\to} \mu, \quad \text{as } N \to \infty.$$

Moreover, by the central limit theorem,

$$\|\tilde{\mu}_N - \mu\|_2 = \sqrt{\mathbb{V}(\tilde{\mu}_N)} = \frac{\mathbb{V}(f(S_T))}{\sqrt{N}} \sim \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).$$

Can a quantum computer do better?

# Encoding the problem

Björn showed us how to construct a unitary operator $\mathcal{U}$ encoding the desired quantity $\mu$ as

$$\mathcal{U}\,|0\rangle_{n+1} = \sqrt{1-a}\,|\psi_0\rangle_n\,|0\rangle + \sqrt{a}\,|\psi_1\rangle_n\,|1\rangle\,.$$

I.e., $a \in [0,1]$ is the probability of measuring $|1\rangle$ in the $(n+1)$-th qubit.

Can we estimate $a$?

# Amplitude estimation

- Quantum Amplitude Estimation (QAE) does just that.
- But first: Phase estimation.

# Phase estimation

Given:

- Unitary operator $\mathcal{U}$,
- Known eigenvector $|\psi\rangle$, ...
- ... with associated **unknown** eigenvalue $\lambda = e^{i2\pi q/2^n}$, $q \in [0, 2^n)$.

Task: Estimate $\lambda$.

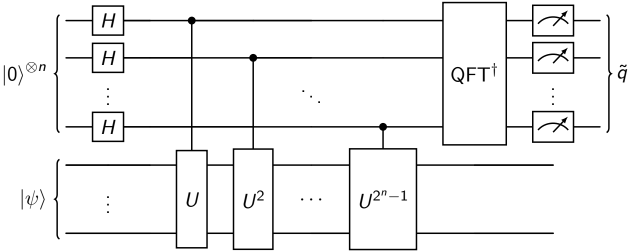



Figure: Phase estimation circuit.

# Phase estimation

**Theorem (Quantum phase estimation)**

*With probability* $1-\delta$*, the estimate* $\tilde{q}$ *provided by QPE satisfies*

$$|\tilde{q}-q| \leq \frac{2}{N}\left(2+\frac{1}{2\delta}\right) \sim \mathcal{O}\left(\frac{1}{N}\right),$$

*where* $N = 2^n$*.*

# Phase estimation

*"Proof idea" (integer case).*
Assume $q \in \{0, 1, ..., 2^n - 1\}$. Then,

$$|0\rangle^{\otimes n} |\psi\rangle \xmapsto{H^{\otimes n}} \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} |j\rangle |\psi\rangle \longmapsto$$

$$\xmapsto{\text{Controlled } \mathcal{U}\text{'s}} \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} |j\rangle e^{2\pi i j q/2^n} |\psi\rangle =$$

$$\overset{\text{Phase kickback}}{=} \left( \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{2\pi i j q/2^n} |j\rangle \right) |\psi\rangle =$$

$$= \mathrm{QFT} |q\rangle |\psi\rangle \xmapsto{\mathrm{QFT}^\dagger} |q\rangle |\psi\rangle .$$

□

# Phase estimation

Quantum Fourier Transform acts as

$$\mathrm{QFT}\,|q\rangle = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{2\pi i j q/2^n}\,|j\rangle\,.$$

Therefore,

$$|0\rangle^{\otimes n}\,|\psi\rangle \overset{\text{Phase estimation}}{\longmapsto} |q\rangle\,|\psi\rangle$$

□

The eigenvalue can be recovered as $\lambda = e^{2\pi i j q/2^n}$.

**Note**. For $q \in \{0, 1, ..., 2^n - 1\}$, phase estimation introduces no error.
**Note'**. For the general case $q \in [0, 2^n)$, see [Kit95], [MW23].

# Amplitude estimation

Given: unitary operator $\mathcal{U}$ acting on $n+1$ qubits as

$$\mathcal{U}\left|0\right\rangle_{n+1} = \sqrt{1-a}\left|\psi_0\right\rangle_n\left|0\right\rangle + \sqrt{a}\left|\psi_1\right\rangle_n\left|1\right\rangle.$$

Task: estimate $a$, i.e., the probability of measuring 1 in the $(n+1)$-th qubit.

# Amplitude estimation

Amplitude estimation uses a combination of phase estimation and amplitude amplification [Bra+02] in order to get an estimate of $a$.

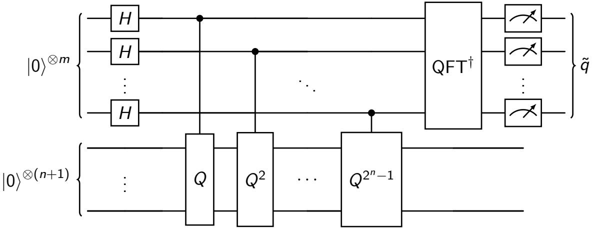



Figure: Amplitude estimation circuit.

Where $\mathcal{Q}$ is the Grover operator

$$\mathcal{Q} = \mathcal{U}\mathcal{S}_0\mathcal{U}^\dagger\mathcal{S}_{\psi_0},$$

and

$$\mathcal{S}_{\psi_0} = \mathbb{I} - 2\,|\psi_0\rangle_n \langle\psi_0|_n \otimes |0\rangle\langle 0|, \quad \mathcal{S}_0 = \mathbb{I} - 2|0\rangle_{n+1}\langle 0|_{n+1}.$$

# Amplitude estimation

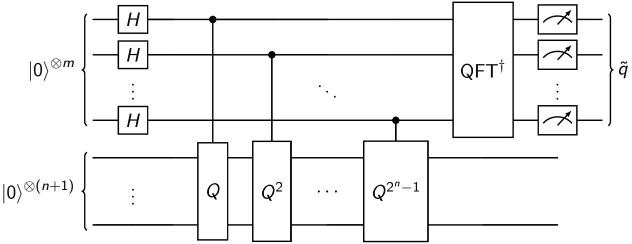



Figure: Amplitude estimation circuit.

The measured number $\tilde{q} \in \{0, 1, ..., 2^m - 1\}$ is then mapped to the estimate

$$\tilde{a} = \sin^2\left(\tilde{\theta}_a\right),$$

where

$$\tilde{\theta}_a = \frac{\pi \tilde{q}}{M}, \quad M = 2^m.$$

# Amplitude estimation

### Theorem (Quantum amplitude estimation)

*With probability* $8/\pi^2 \approx 0.81$, *the estimate* $\tilde{a}$ *provided by QAE satisfies*

$$|\tilde{a} - a| \leq \frac{2\pi\sqrt{a(1-a)}}{M} + \frac{\pi^2}{M^2} \sim \mathcal{O}\left(\frac{1}{M}\right).$$

Quadratic speedup over classical Monte Carlo!

- Monte Carlo: $\mathcal{O}(1/\sqrt{N})$,
- Amplitude estimation: $\mathcal{O}(1/M)$.

In practice: run QAE multiple times and take the median value. This boost success probability close to 100%

# Amplitude estimation

*"Proof idea"*
Use the QPE theorem and the fact that the Grover operator acts as

$$\mathcal{Q}^k \mathcal{U} |0\rangle^{\otimes(n+1)} = \cos((2k+1)\theta_a)|\psi_0\rangle_n |0\rangle + \sin((2k+1)\theta_a)|\psi_1\rangle_n |1\rangle ,$$

which implies

$$\mathbb{P}(|1\rangle) = \sin^2((2k+1)\theta_a),$$

where $\theta_a = \arcsin\left(\sqrt{a}\right)$.

# A toy example

Let $\mathcal{U}$ describe a Bernoulli random variable with success probability $p$:

$$\mathcal{U}|0\rangle = \sqrt{1-p}|0\rangle + \sqrt{p}|1\rangle.$$

We can model $\mathcal{U}$ with a rotation around the $Y$-axis:

$$\mathcal{U} = R_Y(\theta_p), \quad \theta_p = 2\arcsin(\sqrt{p}).$$

The Grover operator for this case is particularly simple:

$$\mathcal{Q} = R_Y(2\theta_p),$$

whose powers are very easy to calculate:

$$\mathcal{Q}^k = R_Y(2k\theta_p).$$

# A toy example

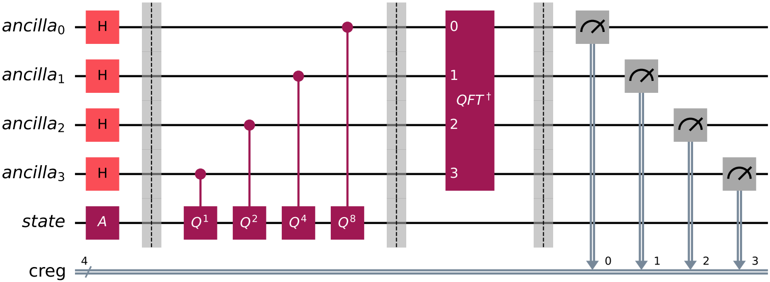



Figure: QAE circuit for binomial random variable, implemented in Qiskit.

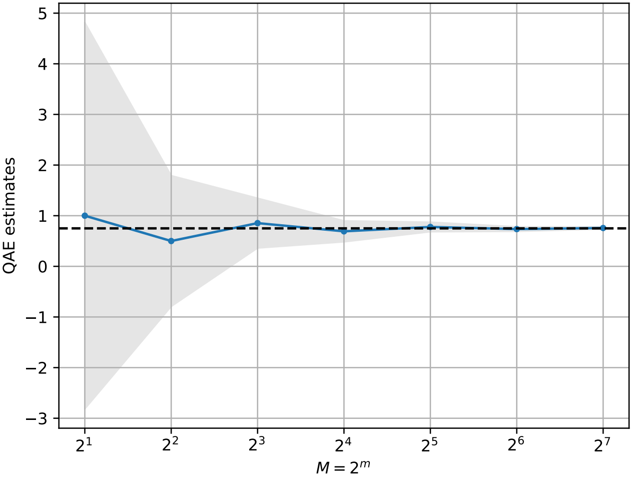


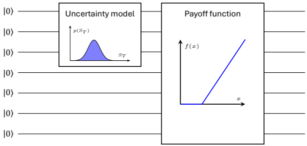

Figure: QAE estimates

# QAE for derivative pricing

We saw how to encode the expected payoff $\mu$ of a european call option into an operator $\mathcal{U}$

$$\mathcal{U}\left|0\right\rangle_{n+1} = \sqrt{1-a}\left|\psi_0\right\rangle_n\left|0\right\rangle + \sqrt{a}\left|\psi_1\right\rangle_n\left|1\right\rangle,$$

after an appropriate re-scaling to the interval $[0, 1]$.




Amplitude estimation can be used to approximate $a$.

# QAE for option pricing

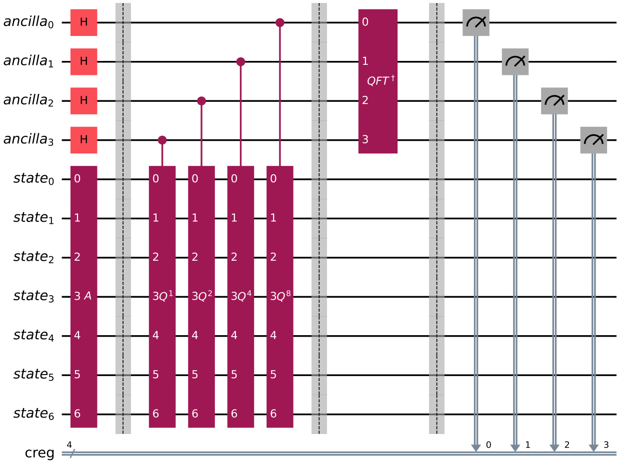



Figure: QAE circuit for option pricing. Implemented in Qiskit.

Notebook time!

# Is QAE NISQ-ready?

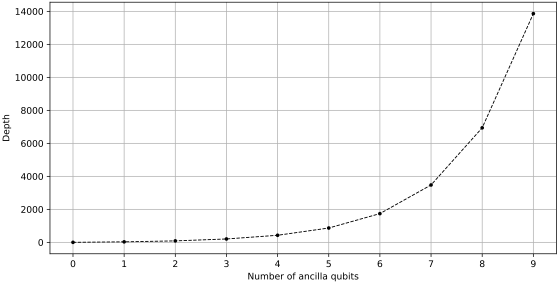



Figure: Depth vs ancillas.

IBM Heron can run $\approx 1800$ gates within the coherence time of its qubits [IBM].

# QAE alternatives

- Iterative quantum amplitude estimation [Gri+21].
- Faster amplitude estimation [Nak20].
- Maximum likelihood amplitude estimation [Suz+20].

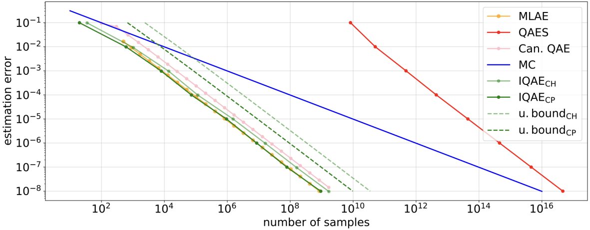



Figure: Convergence rates of different QAE algorithms (Figure 3 in [Gri+21]).

# Conclusions

- Quadratic speedup over classical Monte Carlo,
- Useful in areas of mathematical finance where Monte Carlo is common practice, such as derivative pricing,
- Not NISQ-ready, but active area of research.

# More quantum finance

- [Sta+20]: Portfolios of options, basket options, path-dependent options.
- [WK24]: Asian and barrier options under Heston model.
- [ZLW19]: Option Pricing with qGANs.
- [DL21]: Quantum Support Vector Regression for Disability Insurance.

# References I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