Quantum kernel estimation with application to disability insurance

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Outline

- Disability insurance
- Kernels and support vector regression
- Quantum computers
- Quantum kernel estimation
- Disability insurance model

Introduction

- Health and disability insurance provides economic protection from illness or disability
- Typically, an insured individual receives a monthly payment from an insurance company in the case of illness
- The expected cost should be covered by premium payments
- The insurance company needs to predict future costs using statistical models based on historical data
    - Typically done by estimating transition probabilities between states such as 'healthy', 'ill', 'dead', ...

Disability model

- Consider a population of insured individuals
- Let E_i be the number of healthy individuals from the population subgroup i
- We denote by D_i the number of individuals falling ill amongst the E_i insured healthy individuals:

$$D_i \sim \mathrm{Bin}(E_i, p(x_i))$$

- For each i there is some associated data $x_i \in \mathbb{R}^d$ which may e.g. contain information about age, gender, ...
- $p(x_i)$ is the probability that an individual randomly selected from E_i falls ill

Disability model

- We propose Support Vector Regression (SVR) to model the logistic disability inception probability:

$$\operatorname{logit} p(x) := \log \frac{p(x)}{1 - p(x)} = \sum_{i=1}^{n} \alpha_i K(x, x_i) + \beta$$

- K is a quantum kernel estimated on a quantum computer.
- The parameters $\{\alpha_i\}_i$ and β subsequently fitted using SVR.
- Hybrid quantum-classical learner!
- Functional form guarantees $p(x) \in (0, 1)$.

Review: Kernels and support vector regression

- Let $x_i \in \mathbb{R}^d, y_i \in \mathbb{R}, i = 1, \ldots, n$, be observations in a data set
- A *feature map* $\Phi : \mathbb{R}^d \mapsto \mathcal{F}$ maps a sample data point x to a feature vector $\Phi(x)$ in a feature space $\mathcal{F}$ (Hilbert space with inner product $\langle \cdot, \cdot \rangle$)
- Φ naturally gives rise to a *kernel* through the relation

$$K(x, z) = \langle \Phi(x), \Phi(z) \rangle, \tag{1}$$

- $K(x, z)$ *is a similarity measure* between x and z in the feature space.
- The *reproducing kernel Hilbert space* associated with Φ is defined by

$$\mathcal{R} = \{f : \mathbb{R}^d \mapsto \mathbb{C}; \quad f(x) = \langle w, \Phi(x) \rangle \ \ \forall \ x \in \mathbb{R}^d, w \in \mathcal{F}\}. \tag{2}$$

- $f(x) := \langle w, \Phi(x) \rangle$ can be interpreted as linear models in the feature space $\mathcal{F}$.

Review: Kernels and support vector regression

SVR can be formulated as a convex optimization problem of the form

$$
\begin{aligned}
\mathrm{P:}\quad & \min_{w,b,\xi,\xi'} \ \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}(\xi_i + \xi_i') \\
\text{s.t.}\quad & (w^T\Phi(x_i) + b) - y_i \le \varepsilon - \xi_i, && i = 1, \ldots, n, \\
& y_i - (w^T\Phi(x_i) + b) \le \varepsilon - \xi_i', && i = 1, \ldots, n, \\
& \xi_i,\ \xi_i' \ge 0, && i = 1, \ldots, n,
\end{aligned}
$$

where ε determines the error tolerance of the solution, C is a regularization parameter, and $\xi_i \in \mathbb{R}$ and $\xi_i' \in \mathbb{R}, i = 1, \ldots, n$, are slack variables.

Review: Kernels and support vector regression

The dual formulation D of P is (recall $K(x_i, x_j) = \langle \Phi(x_i), \Phi(x_j) \rangle$)

$$\begin{aligned}
\text{D:} \quad & \max_{\lambda,\lambda'} -\frac{1}{2}\sum_{i,j=1}^{n}(\lambda_i - \lambda_i')(\lambda_j - \lambda_j')K(x_i, x_j) \\
& \quad -\varepsilon\sum_{i=1}^{n}(\lambda_i - \lambda_i') + \sum_{i=1}^{n} y_i(\lambda_i - \lambda_i') \\
\text{s.t.} \quad & \sum_{i=1}^{n}(\lambda_i - \lambda_i') = 0, \\
& 0 \leq \lambda_i \leq C, i = 1, \dots, n, \\
& 0 \leq \lambda_i' \leq C, i = 1, \dots, n,
\end{aligned}$$

The solutions of P and D coincide and are given by

$$f(x) = \sum_{i=1}^{n} \alpha_i K(x, x_i) + \beta, \tag{3}$$

Review: Kernels and support vector regression

- The feature map (and thus the kernel) can be chosen in many different ways
- Ideally, the feature map should be chosen such that the kernel can be efficiently computed
- Well known classical kernels include e.g. the Gaussian kernel:

$$K(x, z) = e^{-\gamma ||x-z||^2}$$

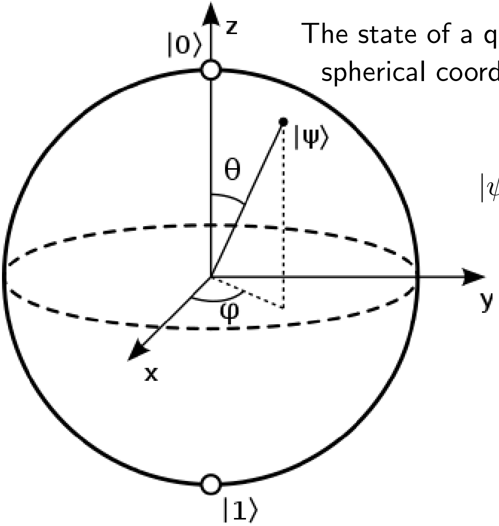
- A modern alternative is provided by the class of quantum kernels
    - Data is mapped to *quantum states* in some *quantum feature (Hilbert) space* $\mathcal{H}$
    - Quantum kernels can be estimated using quantum computers!

Review: Quantum computers

- A quantum computer is a computer that is governed by the laws of quantum physics
- In classical computers, information is represented by bits taking values in $\{0, 1\}$
- Quantum computers use *qubits*
  - Information represented by quantum state

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle\,,\ |a|^2 + |b|^2 = 1.$$

  - A quantum state induces a probability distribution on $\{0, 1\}$
  - At *measurement* of the quantum state of the qubit, an outcome is determined



The state of a qubit can be represented using spherical coordinates on the *Bloch sphere*:

$$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}\,|1\rangle$$

Review: Quantum computers

- Programming a quantum computer with d qubits is performed by creating a *quantum circuit* $\mathcal{A}$
- $\mathcal{A}$ induces a probability measure for a r.v. V on $\{0, 1\}^d$
- Running the circuit $\mathcal{A}$ essentially means sampling from V
- Intuitively appealing to probabilists, statisticians, actuaries, quants, ...

Review: Quantum kernel estimation

- Let $\Phi : x \mapsto \Phi(x)$ be a *quantum feature map* that maps a data point to a quantum state in a Hilbert space $\mathcal{H}$
- Any quantum state $\psi \in \mathcal{H}$ satisfies the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(t,x) = H\psi(t,x), \quad \psi(0,\cdot) \in \mathcal{H} \text{ is given}, \tag{4}$$

  where H is the Hamiltonian operator associated to the quantum system.
- If H is time-independent, the solution to (4) is given by

$$\psi(t,x) = U(t)\psi(0,x), \tag{5}$$

  where the operator U defined by

$$U(t) = e^{-iHt/\hbar} \tag{6}$$

  is the unitary time evolution operator associated with H.

Review: Quantum kernel estimation

- For every pair (Φ, x) there is an operator $U_\Phi(x)$ (*feature embedding circuit*), implicitly determined by

$$\Phi(x) = U_\Phi(x)\Omega_0, \tag{7}$$

where Ω_0 denotes the ground state ($|0\dots0\rangle$).

- Let the kernel K corresponding to Φ be given by

$$K(x,z) = |\langle\Phi(x), \Phi(z)\rangle|^2 = |\Omega_0^\dagger\, U_\Phi^\dagger(z)U_\Phi(x)\,\Omega_0|^2 \tag{8}$$

that is, $K(x,z)$ is given by the probability of obtaining the measurement outcome Ω_0 when measuring the quantum state $\Psi(x,z)$ defined by

$$\Psi(x,z) = U_\Phi^\dagger(z)U_\Phi(x)\Omega_0, \tag{9}$$

Review: Quantum kernel estimation

- The kernel can now be estimated on a quantum computer!
  - We load the state $\Psi(x, z)$ into a quantum circuit.
  - This circuit is run n times
  - $K(x, z)$ is estimated by the frequency of Ω_0-measurements.
- The form (8) of the kernel is what allows us to estimate it using a quantum computer! i.e.

$$K(x, z) = |\langle \Phi(x), \Phi(z) \rangle|^2 = |\Omega_0^\dagger \, U_\Phi^\dagger(z) U_\Phi(x) \, \Omega_0|^2$$

Disability model

- We propose to model the logistic disability inception probability $\operatorname{logit} p(x)$ as

$$\operatorname{logit} p(x) := \log \frac{p(x)}{1 - p(x)} = \sum_{i=1}^{n} \alpha_i K(x, x_i) + \beta,$$

where K is a quantum kernel (to be defined) that is to be estimated on a quantum computer, and the parameters $\{\alpha_i\}_i$ and β are to be subsequently fitted using SVR.

Disability model

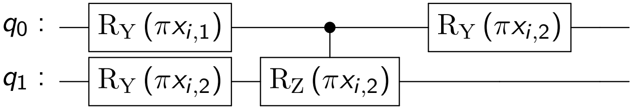
- Our data: gender ($x_{i,1}$) and age ($x_{i,2}$)
- We choose the kernel K associated with the unitary operator $U_\Phi(\cdot)$ defined by

$$U_\Phi(x_i) = \Big(I \otimes \mathrm{R_Y}(\pi x_{i,2})\Big) C_{\mathrm{R_Z}}(\pi x_{i,2}) \Big(\mathrm{R_Y}(\pi x_{i,2}) \otimes \mathrm{R_Y}(\pi x_{i,1})\Big), \tag{10}$$

- $\mathrm{R_Y}(\cdot)$ denotes a rotation around the Y-axis of the Bloch sphere
- $C_{\mathrm{R_Z}}(\cdot)$ denotes a rotation around the Z-axis for the second qubit, conditional on the state of the first qubit.

Disability model

The unitary operator (10) can be represented by the quantum circuit



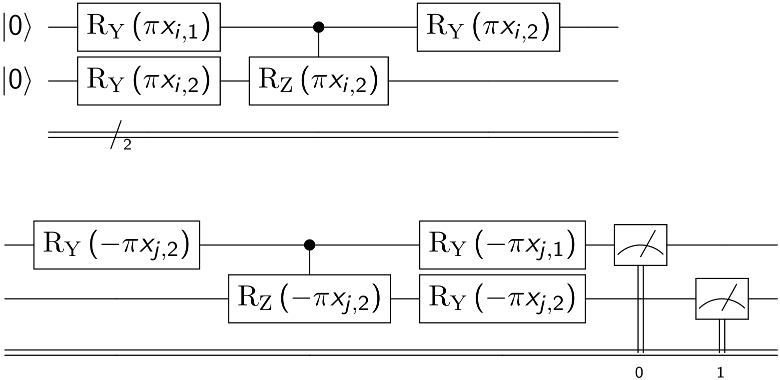
- $x_{i,1}$ takes the value 1 if the population subgroup is male, and 0 otherwise
- $x_{i,2}$ is the age of the population subgroup, in centuries.

This circuit is designed to

- clearly separate male and female subgroups.
- gradually increase the dissimilarity between different age groups as the difference in ages increases.

Disability model

For each pair (x_i, x_j), we run this quantum circuit inserting the values of x_i, and then run the adjoint circuit inserting the values of x_j:



Numerical results: kernel

- We perform simulations on the IBM Yorktown quantum computer
- For each pair (x_i, x_j) we
  - run the circuit 8192 times and measure the outcomes
  - estimate $K(x_i, x_j)$ with the observed frequency of the ground state.
- Binomial sampling error small ($< 1\%$), hardware error dominates
- Results are compared with exact (classically determined) kernel

Numerical results: kernel

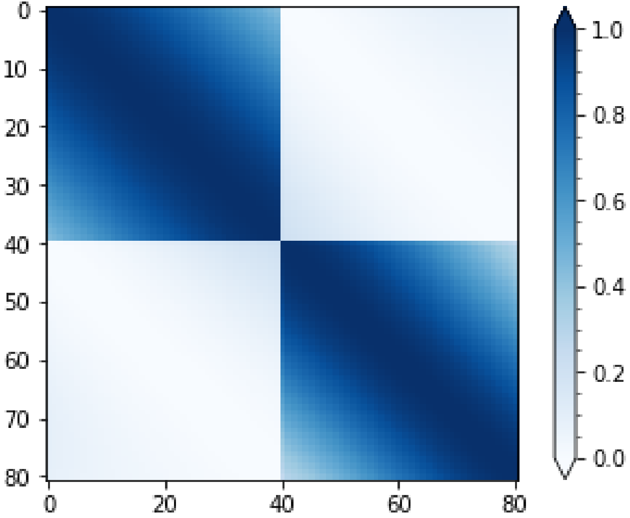


Figure: Classically determined Kernel matrix.

Numerical results: kernel

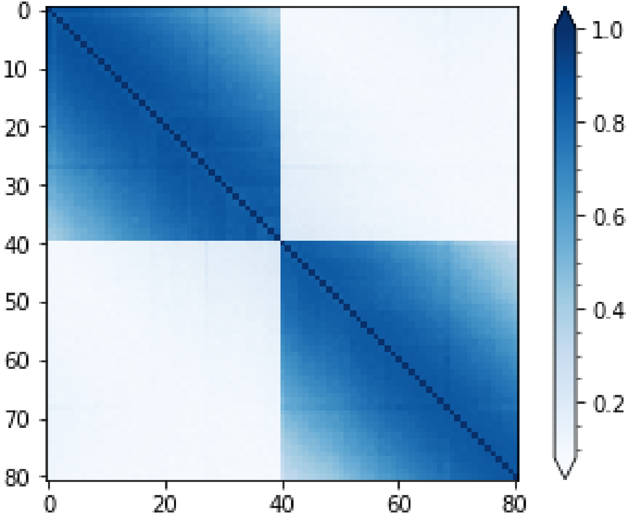


Figure: Kernel matrix estimated on the IBM Yorktown quantum

Numerical results: disability inception

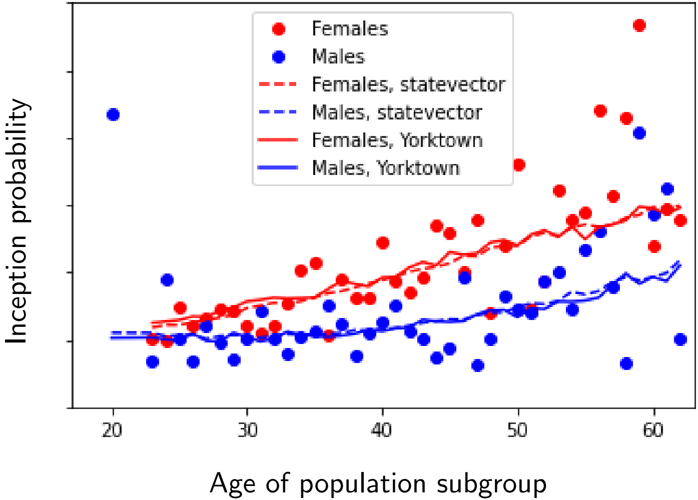


Figure: Out-of-sample disability inception rates estimated by state vector simulation and from the IBM Yorktown quantum computer.

Numerical results: disability inception

Leave-one-out crossvalidation:

Table: Weighted out-of-sample R^2 for the classical and quantum kernels.

kernel	R^2
polynomial	0.550
state vector quantum kernel	0.541
Gaussian kernel	0.529
Yorktown quantum kernel	0.518
sigmoid	0.494
linear	0.426

Conclusions

- We propose a hybrid classical-quantum approach to estimate disability inception probabilities
- Suggested model performs similar to existing classical model, even on noisy hardware
- The approach is not restricted to insurance applications, and can be used for general regression and classification problems, e.g. Credit Risk, Fraud detection, ...
- Outlook: As the hardware improves and becomes more powerful, this approach might be able to surpass classical models

Selected references

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